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AN IMPROVED FORMULATION OF NONLINEAR STRAIN-DISPLACEMENT RELATIONS AND SPECIFIC MATHEMATICAL MODELS FOR STABILITY ANALYSIS OF THIN RECTANGULAR PLATES

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ABSTRACT: *The present work is aimed at deriving from the first principle the strain-displacement relations, and to develop specific stability equations for five (5) plate types using polynomial displacement shape functions for large deflection analysis of thin rectangular plates. This was done by looking at the deformation of a cube and formulating the new nonlinear strain-displacement relations, and substituting the new relations together with stress-strain relations into the total potential energy functional equation to arrive at the general stress parameter equation. From this equation, the general mathematical model for stability analysis of thin rectangular plates was formulated. The polynomial displacement shape function for each of the five plate types considered here was evaluated to obtain the individual stiffness. The obtained stiffnesses were substituted into the general stability model and evaluated to obtain specific mathematical models for the five plate types. The observed numerical values indicate that the frequency increases as the displacement increases and decreases as the aspect ratio increases. This conforms with existing works in literature and the behavior of plates. Therefore, the conclusion is that these newly formulated mathematical models are adequate for these analyses.*

KEYWORDS*:* *Strain-displacement relations, Stress parameter, Buckling and Postbuckling loads, Stability, Large deflection, Specific Models*

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I. INTRODUCTION

A mathematical model is a description of a system using mathematical concepts and language. The process of developing a mathematical model is termed mathematical modeling. Von Karmon's large deflection equations proposed in 1910 are the fundamental equations or models of large deflection analyses of the plates. These equations involved the use of Airy's stress functions. These equations are coupled, non-linear second-degree partial differential equations of fourth order each (Elsheikh & Wang, 2005). Chajes (1974) and Lee (1977) have asserted that a closed-form solution to these equations is not possible. Levy (1942) offered the most acclaimed exact solution to these equations. Levy applied his approach to a plate simply supported on all edges. Several scholars have made attempts to profer solution to the problem of large deflection of thin plates among whom are Stein (1984), Bloom and Coffin (2001), Byklum and Amdahl (2002), and Tanriöver and Senocak (2004). The major challenge of large deflection is the determination of the middle surface displacement. Because beyond the limit of small deflection theory which is based on Kirchoff analysis, the middle surface displacements are no longer negligible. Timoshenko (1959), GhannadPour and Alinia (2006), and Shufrin, Rabinovitch, and Eisenberger (2008), all attempted this problem by assuming the middle surface displacement. Civalek and Yavas (2006) studied statically large deflection of rectangular plates on two parameter-elastic foundations by using the discrete singular convolution (DSC) method on Von Karman's large deflection theory. Their work indicated that increasing the applied loads would increase the displacements. Katsikadelis & Babouskos (2007) used the boundary element method (BEM) based on a variational method for post-buckling analysis of thin elastic plates of arbitrary shape under general boundary conditions. It was concluded from their work that by specifying appropriately the shape of the initial deflection, the bearing capacity can be increased. Oguaghamba (2015), and Oguaghamba, et al (2015), determined the stress functions used in their works and evaluated the buckling and post-buckling stress using the principle of virtual work. Enem (2018) determined the stress functions in his work on the pure bending of thin plates with large deflection. Their attempt covered several plate types, unlike previous works that were limited to the SSSS plate only. However, the approaches were still based on solving the two von Karman equations. Elsami (2018) studied large deflection of plates and stated that von-Karman type nonlinear strain-displacement relations are mostly used in large deflection analysis of rectangular plates. Most of the research works in this area have been around von Karman large deflection equations, and the majority now are based on the use of numerical methods mostly finite element methods.

Based on the observed challenges and the need to proffer a simple theoretical analysis approach to large deflection of thin rectangular plates devoid of Airy's stress functions. Ibearugbulem et al (2020) formulated a new postbuckling equation and applied it to a plate simply supported all-round (SSSS). The present work aims to formulate the strain-displacement relations from the first principle and apply it to formulate specific new mathematical models for predicting the buckling and postbuckling load of five plate types other than SSSS plate using polynomial displacement shape functions. This will provide simple equations and data for analysts and designers of lightweight plated structures easily.

II. MATERIALS AND METHODS

The materials used are the formulated displacement shape function for six plate types as shown in Table 1. The methodology used is presented in the following section.

1. STRAIN-DISPLACEMENT FORMULATION AND THE GENERAL STABILITY EQUATION OF A PLATE UNDER LARGE DEFLECTION

A. MIDDLE SURFACE DISPLACEMENT

The major assumption of the analysis of plates with large deflection is that the middle surface displacements are not zeros. We need to understand the nature of these displacements to determine their values. Consider a cube where point B of line A-B moved to point B' as shown in Fig.1.

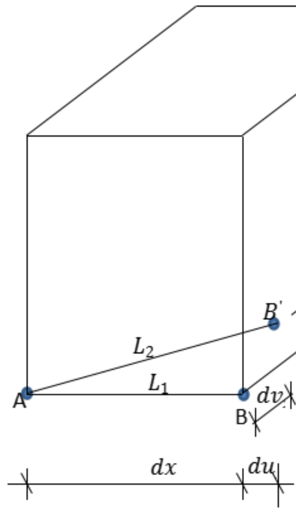


Fig.1: Large displacement of a cube

Large displacements of the sides of the tiny cube are shown in Fig. 1. Some of the notations on the figure are defined as follows:

$$du_x = \frac{\partial u}{\partial x}dx\ ;\ dv_x = \frac{\partial v}{\partial x}dx;\ \ dw_x = \frac{\partial w}{\partial x}dx \qquad (1a,b,c)$$

$$du_y = \frac{\partial u}{\partial y}dy\ ;\ dv_y = \frac{\partial v}{\partial y}dy;\ \ dw_y = \frac{\partial w}{\partial y}dy \qquad (1d,e,f)$$

$$du_z = \frac{\partial u}{\partial z}dz\ ;\ dv_z = \frac{\partial v}{\partial z}dz\ ;d\,w_z = \frac{\partial w}{\partial z}dz \qquad (1.3g,h,i)$$

The length of the side A - B of the cube in x direction before deformation is L_1. It is the same as dx.
After deformation, point B goes to new point, B'. The length A - B' is L_2 and defined by Pythagoras as:

$$L_2 = \sqrt{([dx + d\,u_x]^2 + dv_x^2 + dw_x^2)} \qquad (2)$$

Expanding Equation (2) and substituting Equations (1a,b,c) into the resulting expression gives Equation (3a).

$$L_2 = \sqrt{\left[dx^2 + 2\,\frac{\partial u}{\partial x}dx^2 + \left(\frac{\partial u}{\partial x}\right)^2 dx^2 + \left(\frac{\partial v}{\partial x}\right)^2 dx^2 + \left(\frac{\partial w}{\partial x}\right)^2 dx^2\right]} \qquad (3a)$$

That is, from equation (3a):

$$L_2 = dx\sqrt{\left[1 + 2\,\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]} \qquad (3b)$$

Expanding Equation (3b) using Binomial expansion gives:

$$L_2 = dx\left[1 + \frac{1}{2}\left\{2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right\} + .\ \ .\ \ .\right] \qquad (3c)$$

Approximating Equation (3c) gives:

$$L_2 = dx\left[1 + \frac{1}{2}\left\{2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right\}\right] \quad (3d)$$

The squares of small quantities, du/dx and dv/dy are too small that they can be neglected without grossly affecting the result. Thus, Equation (3d) becomes:

$$L_2 = dx + \frac{\partial u}{\partial x}dx + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 dx \quad (4)$$

Deformation in length, Δ_x is obtained by subtracting L_1 from L_2. That is:

$$\Delta_x = L_2 - L_1 = dx + \frac{\partial u}{\partial x}dx + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 d - dx.$$

That is:

$$\Delta_x = \frac{\partial u}{\partial x}dx + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 dx \quad (5a)$$

Similarly,

$$\Delta_y = \frac{\partial v}{\partial y}dy + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 dy \quad (5b)$$

For thin plates, the engineering vertical shear strains, γ_{xz}, and γ_{yz} are zeros. That is:

$$\gamma_{xz} = \varepsilon_{xz} + \varepsilon_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0 \quad (6a)$$

$$\gamma_{yz} = \varepsilon_{yz} + \varepsilon_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0 \quad (6b)$$

Solving Equations (6a) and (6b) for u and v respectively gives:

$$u = -z\frac{\partial w}{\partial x} + u_0 \quad (7a)$$

$$v = -z\frac{\partial w}{\partial y} + v_0 \quad (7b)$$

Where u_0 and v_0 are constants of integration and represent the middle surface displacements.

Substituting Equations (7a) and (7b) respectively into Equations (5a) and (5b) gives:

$$\Delta_x = \left[-z\frac{\partial^2 w}{\partial x^2} + \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]dx \quad (8a)$$

$$\Delta_y = \left[-z\frac{\partial^2 w}{\partial y^2} + \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]dy \quad (8b)$$

In notation (symbolic) form, Equations (8a) and (8b) can respectively be rewritten as:

$$\Delta_x = \Delta_{xb} + \Delta_{xm} \quad (8c)$$

$$\Delta_y = \Delta_{yb} + \Delta_{ym} \quad (8d)$$

Where:

$$\Delta_{xb} = -z\frac{\partial^2 w}{\partial x^2}dx \quad (8e)$$

$$\Delta_{yb} = -z\frac{\partial^2 w}{\partial y^2}dy \quad (3.8f)$$

$$\Delta_{xm} = \frac{\partial u_0}{\partial x}dx + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 dx \quad (3.8g)$$

$$\Delta_{ym} = \frac{\partial v_0}{\partial y}dy + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 dy \quad (3.8h)$$

Equations (8g) and (8h) are the membrane in-plane displacements and are not functions of plate thickness. These displacements are assumed to be zero when the plate is under small displacement. Thus, minimizing them with respect to dx and dy respectively results to:

$$\frac{\partial u_0}{\partial x} = -\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \quad (9a)$$

$$\frac{\partial v_0}{\partial y} = -\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \quad (9b)$$

However, if the displacement is not small (that is large displacement) then Equations (9a) and (9b) are rewritten by replacing "minus half" with another constant c_1. That is:

$$\frac{\partial u_0}{\partial x} = c_1\left(\frac{\partial w}{\partial x}\right)^2 \quad (10a)$$

$$\frac{\partial v_0}{\partial y} = c_1\left(\frac{\partial w}{\partial y}\right)^2 \quad (10b)$$

Note that whenever c_1 becomes exactly equal to "minus one" the plate goes into small displacement. Substituting Equations (10a) and (10b) respectively into Equations (8a) and (8) results to:

$$\Delta_x = \left[-z\frac{\partial^2 w}{\partial x^2} + \left[c_1 + \frac{1}{2}\right]\left(\frac{\partial w}{\partial x}\right)^2\right]dx \quad (11a)$$

$$\Delta_y = \left[-z\frac{\partial^2 w}{\partial y^2} + \left[c_1 + \frac{1}{2}\right]\left(\frac{\partial w}{\partial y}\right)^2\right]dy \quad (11b)$$

B. MEMBRANE STRAINS

Membrane strains are strains developed in a plate when it resists load by membrane action. The normal strain in the x direction is the ratio of change in length to the original length. That is:

$$\varepsilon_{xx} = \frac{\Delta_x}{dx} \quad (12a)$$

$$\varepsilon_{yy} = \frac{\Delta_y}{dy} \quad (12b)$$

Similarly, the engineering in-plane shear strain is defined as:

$$\gamma_{xy} = \frac{\Delta_x}{dy} + \frac{\Delta_y}{dx} \quad (12c)$$

Substituting Equations (11a) and (11b) respectively into Equations (12a) and (12b) results to:

$$\varepsilon_{xx} = -z\frac{\partial^2 w}{\partial x^2} + c_2\left(\frac{\partial w}{\partial x}\right)^2 \quad (13a)$$

$$\varepsilon_{yy} = -z\frac{\partial^2 w}{\partial y^2} + c_2\left(\frac{\partial w}{\partial y}\right)^2 \quad (13b)$$

Where:

$$c_2 = c_1 + \frac{1}{2} \quad (3.13c)$$

Substituting Equations (11a) and (11b) into Equations (12c) results to:

$$\gamma_{xy} \cong -z\frac{\partial^2 w}{\partial x \partial y} + \left[c_1 + \frac{1}{2}\right]\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - z\frac{\partial^2 w}{\partial x \partial y} + \left[c_1 + \frac{1}{2}\right]\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

That is:

$$\gamma_{xy} = 2\left(-z\frac{\partial^2 w}{\partial x \partial y} + c_2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \quad (13d)$$

Minimizing Equation (13a) with respect to w gives:

$$\frac{\partial \varepsilon_{xx}}{\partial w} = -z\frac{\partial^2}{\partial x^2} + 2c_2\frac{\partial^2 w}{\partial x^2} = 0 \quad (14)$$

Factorizing Equation (14) gives:

$$\frac{\partial^2}{\partial x^2}(-z + 2c_2 w) = 0 \quad (15)$$

Two possibilities for a second derivative to be zero are for the number or function differentiated to be constant or zero. Assume that the function of Equation (15) is equal to zero. That is:

$$c_2 = \frac{z}{2w} \quad (16)$$

The maximum strain occurs at the outermost fiber where z = t/2. Thus:

$$c_2 = \frac{\frac{t}{2}}{2w} = \frac{1}{4}.\frac{t}{w} \quad (17)$$

Assuming here that the maximum strain occurs for a stable plate when t/w is not less than unity. Assume that the limiting value for t/w is 1. Thus:

$$c_2 = \frac{1}{4} \quad (18)$$

Substituting Equation (18) into Equation (13c) gives:

$$\frac{1}{4} = c_1 + \frac{1}{2}$$. That is:

$$c_1 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \quad (19)$$

C. NONLINEAR IN-PLANE DISPLACEMENTS

Nonlinear in-plane displacements, (Δ_x, Δ_y), are displacements that occur at the inelastic range of a structural material when subject to external in-plane force. Substituting Equation (19) into Equations (11a) and (11b) gives

$$\Delta_x = \left[-z\frac{\partial^2 w}{\partial x^2} + \frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^2\right]dx \quad (20)$$

$$\Delta_y = \left[-z\frac{\partial^2 w}{\partial y^2} + \frac{1}{4}\left(\frac{\partial w}{\partial y}\right)^2\right]dy \quad (21)$$

D. NONLINEAR STRAIN-DISPLACEMENT RELATIONS

Nonlinear strain-displacement relations are expressions for strains at the inelastic range in terms of displacements. Substituting Equation (18) into Equations (13a), (13b), and (13d) results to:

$$\varepsilon_{xx} = -z\frac{\partial^2 w}{\partial x^2} + \frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^2 \qquad (22)$$

$$\varepsilon_{yy} = -z\frac{\partial^2 w}{\partial y^2} + \frac{1}{4}\left(\frac{\partial w}{\partial y}\right)^2 \qquad (23)$$

$$\gamma_{xy} = 2\left(-z\frac{\partial^2 w}{\partial x \partial y} + \frac{1}{4}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \qquad (24)$$

Equations (22) to (24) is the formulated strain-displacement relations. Next is to use them for stability equation formulation.

E. TOTAL POTENTIAL ENERGY FUNCTIONAL

The total potential energy, Π, of a thin rectangular plate is given (Ibearugbulem, 2017)

$$\Pi = \frac{1}{2}\int_0^a\int_0^b\int_{\frac{-t}{2}}^{\frac{t}{2}} \left[\sigma_{xx}\,\varepsilon_{xx} + \sigma_{yy}\,\varepsilon_{yy} + \tau_{xy}\gamma_{xy}\right] dxdydz - \frac{N_x}{2}\int_0^a\int_0^b \left[\frac{\partial w}{\partial x}\right]^2 dxdy \qquad (25)$$

And, the constitutive Relations are given as:

$$\sigma_x = \frac{E}{1-\upsilon^2}\left(\varepsilon_{xx} + \upsilon\varepsilon_{yy}\right) \qquad (26a)$$

$$\sigma_y = \frac{E}{1-\upsilon^2}\left(\varepsilon_{yy} + \upsilon\varepsilon_{xx}\right) \qquad (26b)$$

$$\tau_{xy} = \frac{E(1-\upsilon)}{2(1-\upsilon^2)}\gamma_{xy} \qquad (26c)$$

The Equations (26) will be substituted into Equation (25)

$$\Pi = \frac{1}{2}\int_0^a\int_0^b\int_0^z \left[\frac{E}{1-\upsilon^2}\left(\varepsilon_{xx} + v\varepsilon_{yy}\right)\varepsilon_{xx} + \frac{E}{1-\upsilon^2}\left(\varepsilon_{yy} + v\varepsilon_{xx}\right)\varepsilon_{yy} + \frac{E(1-\upsilon)}{2(1-\upsilon^2)}\gamma_{xy}\gamma_{xy}\right] dxdydz - \frac{N_x}{2}\int_0^a\int_0^b \left(\frac{\partial w}{\partial x}\right)^2 dxdy$$

Simplifying yields

$$\Pi = \frac{E}{2(1-\upsilon^2)}\int_0^a\int_0^b\int_0^z \left[\varepsilon_{xx}^2 + 2\upsilon\varepsilon_{xx}\,\varepsilon_{yy} + \frac{\gamma_{xy}^2}{2} - \upsilon\frac{\gamma_{xy}^2}{2} + \varepsilon_{yy}^2\right] dxdydz - \frac{N_x}{2}\int_0^a\int_0^b \left(\frac{\partial w}{\partial x}\right)^2 dxdy \qquad (27)$$

Substituting Equations (22), (23), and (24) into Equation (27) yields

$$\Pi = \frac{E}{2(1-\upsilon^2)} \int_0^a\int_0^b\int_0^z \{\left[-z\frac{\partial^2 w}{\partial x^2}+\frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^2\right]^2 + 2\upsilon\left[-z\frac{\partial^2 w}{\partial x^2}+\frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^2\right]\left[-z\frac{\partial^2 w}{\partial y^2}+\frac{1}{4}\left(\frac{\partial w}{\partial y}\right)^2\right]$$
$$+2\left[-z\frac{\partial^2 w}{\partial x\partial y}+\frac{1}{4}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\right]^2 - 2\upsilon\left[-z\frac{\partial^2 w}{\partial x\partial y}+\frac{1}{4}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\right]^2$$
$$+\left[-z\frac{\partial^2 w}{\partial y^2}+\frac{1}{4}\left(\frac{\partial w}{\partial y}\right)^2\right]^2\}\, dxdydz \; - \frac{N_x}{2}\int_0^a\int_0^b\left(\frac{\partial w}{\partial x}\right)^2 dxdy \qquad (27a)$$

Expanding the square brackets yields Equation (27b)

$$\Pi = \frac{E}{2(1-\upsilon^2)} \int_0^a\int_0^b\int_{-\frac{t}{2}}^{\frac{t}{2}} \{z^2\left(\frac{\partial^2 w}{\partial x^2}\right)^2 - \frac{2z}{4}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{16}\left(\frac{\partial w}{\partial x}\right)^4$$
$$+2\upsilon\left[z^2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{z}{4}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial y}\right)^2 - \frac{z}{4}\frac{\partial^2 w}{\partial y^2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{16}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2\right]$$
$$+2\left[z^2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{2z}{4}\frac{\partial^2 w}{\partial x\partial y}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) + \frac{1}{16}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2\right]$$
$$-2\upsilon\left[z^2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{2z}{4}\frac{\partial^2 w}{\partial x\partial y}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) + \frac{1}{16}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2\right] + z^2\left(\frac{\partial^2 w}{\partial y^2}\right)^2 - \frac{2z}{4}\frac{\partial^2 w}{\partial y^2}\left(\frac{\partial w}{\partial y}\right)^2$$
$$+\frac{1}{16}\left(\frac{\partial w}{\partial y}\right)^4\}dxdydz$$
$$-\frac{N_x}{2}\int_0^a\int_0^b\left(\frac{\partial w}{\partial x}\right)^2 dxdy \qquad (27b)$$

Opening up the square brackets by multiplying out term yields

$$\Pi = \frac{E}{2(1-\upsilon^2)} \int_0^a\int_0^b\int_{-\frac{t}{2}}^{\frac{t}{2}} \{z^2\left(\frac{\partial^2 w}{\partial x^2}\right)^2 - \frac{z}{2}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{16}\left(\frac{\partial w}{\partial x}\right)^4 + 2\upsilon z^2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\upsilon z}{2}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial y}\right)^2$$
$$-\frac{\upsilon z}{2}\frac{\partial^2 w}{\partial y^2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\upsilon}{8}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2 + 2z^2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - z\frac{\partial^2 w}{\partial x\partial y}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) + \frac{1}{8}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2$$
$$-2\upsilon z^2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 + \upsilon z\frac{\partial^2 w}{\partial x\partial y}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) - \frac{v}{8}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2 + z^2\left(\frac{\partial^2 w}{\partial y^2}\right)^2 - \frac{z}{2}\frac{\partial^2 w}{\partial y^2}\left(\frac{\partial w}{\partial y}\right)^2$$
$$+\frac{1}{16}\left(\frac{\partial w}{\partial y}\right)^4\}dxdydz - \frac{N_x}{2}\int_0^a\int_0^b\left(\frac{\partial w}{\partial x}\right)^2 dxdy \qquad (27c)$$

Collecting like terms yields Equation (27d)

$$\Pi = \frac{E}{2(1-\upsilon^2)} \int_0^a\int_0^b\int_{-\frac{t}{2}}^{\frac{t}{2}} \{z^2[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\upsilon\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 + 2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - 2\upsilon\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2]$$
$$-\frac{z}{2}[\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2 + \upsilon\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial y}\right)^2 + \upsilon\frac{\partial^2 w}{\partial y^2}\left(\frac{\partial w}{\partial x}\right)^2 + 2\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial y}\right)^2 - 2\upsilon\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial y}\right)^2$$
$$+\frac{\partial^2 w}{\partial y^2}\left(\frac{\partial w}{\partial y}\right)^2] + \frac{1}{16}[\left(\frac{\partial w}{\partial x}\right)^4 + 2\upsilon\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2 - 2\upsilon\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2$$
$$+\left(\frac{\partial w}{\partial y}\right)^4]\}dxdydz - \frac{N_x}{2}\int_0^a\int_0^b\left(\frac{\partial w}{\partial x}\right)^2 dxdy \qquad (27d)$$

Canceling out yields, Equation (27d) reduces to Equation (27e)

$$\Pi = \frac{E}{2(1-\upsilon^2)}\int_0^a\int_0^b\int_{-\frac{t}{2}}^{\frac{t}{2}} z^2\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] dxdydz$$
$$- \frac{E}{4(1-\upsilon^2)}\int_0^a\int_0^b\int_{-\frac{t}{2}}^{\frac{t}{2}} z\left[\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2 + 2\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial w}{\partial y}\right)^2\right] dxdydz$$
$$- \frac{E}{2(1-\upsilon^2)}\int_0^a\int_0^b\int_{-\frac{t}{2}}^{\frac{t}{2}}\left[\left(\frac{\partial w}{\partial x}\right)^4 + 2\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^4\right] dxdydz$$
$$- \frac{N_x}{2}\int_0^a\int_0^b\left(\frac{\partial w}{\partial x}\right)^2 dxdy \qquad (27e)$$

After integrating one Equation (27e) with respect to z we have Equation (28)

$$\Pi = \frac{Et^3}{2*12(1-\upsilon^2)}\int_0^a\int_0^b\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] dxdy$$
$$- \frac{Et}{2*16(1-\upsilon^2)}\int_0^a\int_0^b\left[\left(\frac{\partial w}{\partial x}\right)^4 + 2\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^4\right]$$
$$- \frac{N_x}{2}\int_0^a\int_0^b\left(\frac{\partial w}{\partial x}\right)^2 dxdy \qquad (28)$$

But $D = \frac{Et^3}{12(1-\upsilon^2)}$; $g = \frac{12}{t^2}$; (29a,)

$gD = \frac{Et}{(1-\upsilon^2)}$ (29c)

Substituting Equation (29) into Equation (28) yields

$$\Pi = \frac{D}{2}\int_0^a\int_0^b\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] dxdy$$
$$+ \frac{gD}{2*16}\int_0^a\int_0^b\left[\left(\frac{\partial w}{\partial x}\right)^4 + 2\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^4\right] dxdy$$
$$- \frac{N_x}{2}\int_0^a\int_0^b\left(\frac{\partial w}{\partial x}\right)^2 dxdy \qquad (30)$$

In non-dimensional parameters,

$x = aR,\ \ y = bQ,\ \ 0 \le R \le 1, 0 \le R \le 1$ (31)

Substituting Equation (31) into Equation (30) gives

$$\Pi = \frac{bD}{2a^3}\int_0^1\int_0^1\left[\left(\frac{\partial^2 w}{\partial R^2}\right)^2 + \frac{2}{\mathcal{Z}^2}\left(\frac{\partial^2 w}{\partial R \partial Q}\right)^2 + \frac{1}{\mathcal{Z}^4}\left(\frac{\partial^2 w}{\partial Q^2}\right)^2\right] dRdQ$$
$$+ \frac{gD}{2a^3*16}\int_0^a\int_0^b\left[\left(\frac{\partial w}{\partial R}\right)^4 + \frac{2}{\mathcal{Z}^2}\left(\frac{\partial w}{\partial R}\right)^2\left(\frac{\partial w}{\partial Q}\right)^2 + \frac{1}{\mathcal{Z}^4}\left(\frac{\partial w}{\partial Q}\right)^4\right] dRdQ$$
$$- \frac{\mathcal{Z}N_x}{2}\int_0^1\int_0^1\left(\frac{\partial w}{\partial R}\right)^2 dRdQ \qquad (32)$$

Where aspect ratio $\mathcal{Z} = \frac{b}{a}$ (33)

The approximate and non-intractable solution of a rectangular thin plate in the polynomial form is given (Ibearubulem, 2017).

$$w = A(a_0 + a_1 R + \frac{a_2}{2!}R^2 + \frac{a_3}{3!}R^3 + \frac{a_4}{4!}R^4)(b_0 + b_1 Q + \frac{b_2}{2!}Q^2 + \frac{b_3}{3!}Q^3 + \frac{b_4}{4!}Q^4) \quad (44)$$

$$w = a_i h_x \times b_i h_y = Ah \quad (46a)$$

And at the point of maximum deflection is written as

$$w_{max} = A h_{max} \quad (46b)$$

$$\text{where}, h = (a_0 + a_1 R + \frac{a_2}{2!}R^2 + \frac{a_3}{3!}R^3 + \frac{a_4}{4!}R^4)(b_0 + b_1 Q + \frac{b_2}{2!}Q^2 + \frac{b_3}{3!}Q^3 + \frac{b_4}{4!}Q^4) \quad (47)$$

Substitute Equation (46) into Equation (32) yields

$$\Pi = \frac{bDA^2}{2a^3}\int_0^1\int_0^1\left[\left(\frac{\partial^2 h}{\partial R^2}\right)^2 + \frac{2}{Ƨ^2}\left(\frac{\partial^2 h}{\partial R \partial Q}\right)^2 + \frac{1}{Ƨ^4}\left(\frac{\partial^2 h}{\partial Q^2}\right)^2\right] dRdQ + \frac{bgDA^4}{2a^3 * 16}\int_0^1\int_0^1\left[\left(\frac{\partial h}{\partial R}\right)^4 + \frac{2}{Ƨ^2}\left(\frac{\partial h}{\partial R}\right)^2\left(\frac{\partial h}{\partial Q}\right)^2 + \frac{1}{Ƨ^4}\left(\frac{\partial h}{\partial Q}\right)^4\right] dR - \frac{ƧN_x A^2}{2}\int_0^1\int_0^1\left(\frac{\partial h}{\partial R}\right)^2 dRdQ \quad (48)$$

Minimizing Equation (48) with respect to A yields

$$\frac{\partial \Pi}{\partial A} = \frac{bDA}{a^3}\int_0^1\int_0^1\left[\left(\frac{\partial^2 h}{\partial R^2}\right)^2 + \frac{2}{Ƨ^2}\left(\frac{\partial^2 h}{\partial R \partial Q}\right)^2 + \frac{1}{Ƨ^4}\left(\frac{\partial^2 h}{\partial Q^2}\right)^2\right] dRdQ + \frac{bgDA^3}{8a^3}\int_0^1\int_0^1\left[\left(\frac{\partial h}{\partial R}\right)^4 + \frac{2}{Ƨ^2}\left(\frac{\partial h}{\partial R}\right)^2\left(\frac{\partial h}{\partial Q}\right)^2 + \frac{1}{Ƨ^4}\left(\frac{\partial h}{\partial Q}\right)^4\right] dRdQ - ƧN_x A\int_0^1\int_0^1\left(\frac{\partial h}{\partial R}\right)^2 dRdQ = 0 \quad (49)$$

Multiply Equation (56a) by $\frac{a^3}{b}$

$$D\int_0^1\int_0^1\left[\left(\frac{\partial^2 h}{\partial R^2}\right)^2 + \frac{2}{Ƨ^2}\left(\frac{\partial^2 h}{\partial R \partial Q}\right)^2 + \frac{1}{Ƨ^4}\left(\frac{\partial^2 h}{\partial Q^2}\right)^2\right] dRdQ + \frac{gDA^2}{8}\int_0^1\int_0^1\left[\left(\frac{\partial h}{\partial R}\right)^4 + \frac{2}{Ƨ^2}\left(\frac{\partial h}{\partial R}\right)^2\left(\frac{\partial h}{\partial Q}\right)^2 + \frac{1}{Ƨ^4}\left(\frac{\partial h}{\partial Q}\right)^4\right] dRdQ - N_x a^2\int_0^1\int_0^1\left(\frac{\partial h}{\partial R}\right)^2 dRdQ = 0 \quad (50)$$

Dividing all through by D yields

$$\int_0^1\int_0^1\left[\left(\frac{\partial^2 h}{\partial R^2}\right)^2 + \frac{2}{Ƨ^2}\left(\frac{\partial^2 h}{\partial R \partial Q}\right)^2 + \frac{1}{Ƨ^4}\left(\frac{\partial^2 h}{\partial Q^2}\right)^2\right] dRdQ + \frac{gA^2}{8}\int_0^1\int_0^1\left[\left(\frac{\partial h}{\partial R}\right)^4 + \frac{2}{Ƨ^2}\left(\frac{\partial h}{\partial R}\right)^2\left(\frac{\partial h}{\partial Q}\right)^2 + \frac{1}{Ƨ^4}\left(\frac{\partial h}{\partial Q}\right)^4\right] dRdQ - N_x\frac{a^2}{D}\int_0^1\int_0^1\left(\frac{\partial h}{\partial R}\right)^2 dRdQ = 0 \quad (51)$$

Equation (51) can be rewritten as;

$$\left[k_{bx} + \frac{2k_{bxy}}{Ƨ^2} + \frac{k_{by}}{Ƨ^4}\right] + \frac{gA^2}{8}\left[k_{mx} + \frac{2k_{mxy}}{Ƨ^2} + \frac{k_{my}}{Ƨ^4}\right] = N_x\frac{a^2}{D}k_{Nx} \quad (52)$$

Where

$$k_{bx} = \int_0^1 \left(\frac{\partial^2 h_x}{\partial R^2}\right)^2 \partial R * \int_0^1 h_y{}^2 \partial Q \qquad (53a)$$

$$k_{bxy} = \int_0^1 \left(\frac{\partial h_x}{\partial R}\right)^2 \partial R * \int_0^1 \left(\frac{\partial h_y}{\partial Q}\right)^2 \partial Q \qquad (53b)$$

$$k_{by} = \int_0^1 h_x{}^2 \partial R * \int_0^1 \left(\frac{\partial^2 h_y}{\partial Q^2}\right)^2 \partial Q \qquad (53c)$$

$$k_{mx} = \int_0^1 \left(\frac{\partial h_x}{\partial R}\right)^4 \partial R * \int_0^1 h_y{}^4 \partial Q \qquad (53d)$$

$$k_{mxy} = \int_0^1 \left(\frac{\partial h_x}{\partial R}\right)^2 . h_x{}^2 \partial R * \int_0^1 \left(\frac{\partial^2 h_y}{\partial Q^2}\right)^2 . h_y{}^2 \partial Q \qquad (53e)$$

$$k_{my} = \int_0^1 h_x{}^4 \partial R * \int_0^1 \left(\frac{\partial^2 h_y}{\partial Q^2}\right)^4 \partial Q \qquad (53f)$$

$$k_{Nx} = \int_0^1 \left(\frac{\partial h_x}{\partial R}\right)^2 \partial R * \int_0^1 h_y{}^2 \partial Q \qquad (53g)$$

Where subscripts b and m represent bending and membrane component respectively.
From Equation (52)

$$K_{bT} + \frac{gA^2}{8} K_{mT} = N_x \frac{a^2}{D} k_{Nx} \qquad (54)$$

where

$$K_{bT} = \left[k_{bx} + \frac{2k_{bxy}}{\varepsilon^2} + \frac{k_{by}}{\varepsilon^4}\right] \qquad (55a)$$

$$K_{mT} = \left[k_{mx} + \frac{2k_{mxy}}{\varepsilon^2} + \frac{k_{my}}{\varepsilon^4}\right] \qquad (55b)$$

Where K_{bT} and K_{mT} are the total bending stiffness and total membrane stiffness respectively and K_{Nx} is the external buckling load stiffness.
Substitute Equation (29) into Equation (54) yields

$$K_{bT} + \frac{12}{8}\left(\frac{A}{t}\right)^2 K_{mT} = \frac{N_x\, a^2}{t} \frac{12(1-\upsilon^2)}{Et^2} k_{Nx} \qquad (56)$$

$$\frac{\sigma_x\, a^2}{Et^2} 12(1-\upsilon^2) k_{Nx} = K_{bT} + \frac{12}{8}\left(\frac{A}{t}\right)^2 K_{mT} \qquad (57)$$

Where

$$\sigma_x = \frac{N_x}{t} \qquad (58)$$

$$\frac{\sigma_x\, a^2}{Et^2} = \frac{1}{12(1-\upsilon^2)k_{Nx}}\left[K_{bT} + \frac{3}{2}\left(\frac{A}{t}\right)^2 K_{mT}\right] \qquad (59)$$

Equation (59) is the Postbuckling Load Parameter.

From Equation (59)

$$\sigma_x = \left\{\frac{1}{12(1-\upsilon^2)k_{Nx}}\left[K_{bT} + \frac{3}{2}\left(\frac{A}{t}\right)^2 K_{mT}\right]\right\} * \frac{Et^2}{a^2} \qquad (60)$$

Therefore, simplifying Equations (60) becomes

$$\sigma_x = \left[\frac{K_{bT}}{k_{Nx}} + \frac{3}{2}\frac{K_{mT}}{k_{Nx}}\left(\frac{A}{t}\right)^2\right]\frac{D}{ta^2} \qquad (61a)$$

$$\sigma_x = \eta \frac{D}{ta^2} \qquad (61b)$$

Where

$\eta = \left[\frac{K_{bT}}{k_{Nx}} + \frac{3}{2}\frac{K_{mT}}{k_{Nx}}\left(\frac{A}{t}\right)^2\right]$ is the postbuckling stress coefficient

$$\text{Then, } N_x = \left[\frac{K_{bT}}{k_{Nx}} + \frac{3}{2}\frac{K_{mT}}{k_{Nx}}\left(\frac{A}{t}\right)^2\right]\frac{D}{a^2} \qquad (62a)$$

$$N_x = \eta\frac{D}{a^2} \qquad (62b)$$

$$\sigma_x = \sigma_{cr} \left(\text{Critical Stress for } \frac{w}{t} = 0\right)$$

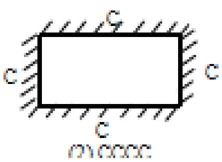
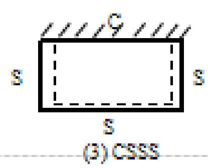
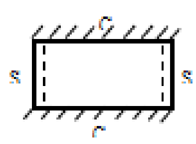
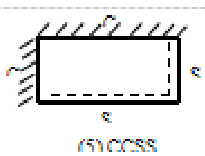
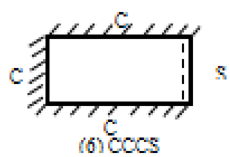
$$\sigma_x = \sigma_{pbx} \left(\text{Postbuckling Stress for } \frac{w}{t} > 0\right)$$

Equations 61 and 62 are the general stability equations for large deflection analysis of a thin rectangular plate for any plate type in terms of stress and load respectively. This equation replaced the long-standing von Karman large deflection equations which have posed a lot of difficulties in proffering solutions to them because they involved stress functions and are coupled nonlinear partial differential equations.

F. NUMERICAL APPLICATION TO FIVE PLATES TYPES USING POLYNOMIAL DISPLACEMENT SHAPE FUNCTIONS.

The polynomial displacement shape profiles, h, are presented in Table 1 for the five plate types under consideration. These profiles were evaluated based on equations 53a-f to obtain the plates' stiffness values ($k_{bx}, k_{bxy}, k_{by}, k_{mx}, k_{mxy}, k_{my}, and\ k_{Nx}$) as presented in Table 2.

Table 1: Plate types, shapes Parameters for Six Plate Types

SN	PLATE BCs	SKETCHES (Plan View)	SHAPE PARAMETER (h) W = Ah; (i.e h = R strip x Q strip)
1	CCCC = (C-C)$_R$ x (C-C)$_Q$		$(R^2 - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4)$
2	CSSS = (S-S)$_R$ x (C-S)$_Q$		$(R - 2R^3 + R^4)(1.5Q^2 - 2.5Q^3 + Q^4)$
3	CSCS = (S-S)$_R$ x (C-C)$_Q$		$(R - 2R^3 + R^4)(Q^2 - 2Q^3 + Q^4)$
4	CCSS = (C-S)$_R$ x (C-S)$_Q$		$(1.5R^2 - 2.5R^3 + R^4)(1.5Q^2 - 2.5Q^3 + Q^4)$
5	CCCS = (C-S)$_R$ x (C-C)$_Q$		$(1.5R^2 - 2.5R^3 + R^4)(Q^2 - 2Q^3 + Q^4)$

These individual plate stiffnesses were thereafter substituted into equations 55a and 55b to obtain the total bending stiffness, k_{bT}, and the membrane stiffness, k_{mT} equations respectively of the various plate types as presented in Tables 3 and 4 respectively. Substituting the total bending stiffness equation and the total membrane stiffness equation for each of the plate types into equations 61 and 62 yields the mathematical models for stability analysis of the various plate types under consideration for any aspect ratio as presented in Table 5.

IV RESULTS AND DISCUSSION

The various results obtained from this study are presented in the various tables as stated in the preceding section.

Table 2: Summary of Stiffness Values for the Six Plate Types

Plate BCs	k_{bx}	k_{bxy}	k_{by}	k_{mx}	k_{mxy}	k_{my}	k_{Nx}
CCCC	0.0012698	0.0003628	0.001269841	0.000000002	0.0000000005	0.0000000024	0.0000302343
CSSS	0.0361905	0.0416327	0.08857143	0.000034098	0.00000457	0.000044925	0.0036621315
CSCS	0.0076190	0.0092517	0.039365079	0.000001645	0.00000026	0.000001925	0.0007709751
CCSS	0.0135714	0.0073469	0.013571429	0.000001179	0.00000015	0.000001179	0.0006462585
CCCS	0.0028571	0.0016327	0.006031746	0.000000057	0.0000000086	0.0000000505	0.0001360544

Table 3: Total Bending Stiffness K_{bT} Equation for Six Plate Types

Plate Type	$K_{bT} = \left[k_{bx} + \frac{2k_{bxy}}{z^2} + \frac{k_{by}}{z^4}\right];$
CCCC	$K_{bT} = \left[0.0012698413 + \frac{0.0007256236}{z^2} + \frac{0.0012698413}{z^4}\right]$
CSSS	$K_{bT} = \left[0.0361904762 + \frac{0.0832653061}{z^2} + \frac{0.0885714286}{z^4}\right]$
CSCS	$K_{bT} = \left[0.0076190476 + \frac{0.0185034014}{z^2} + \frac{0.0393650794}{z^4}\right]$
CCSS	$K_{bT} = \left[0.0135714286 + \frac{0.0146938776}{z^2} + \frac{0.0135714286}{z^4}\right]$
CCCS	$K_{bT} = \left[0.0028571429 + \frac{0.0032653061}{z^2} + \frac{0.0060317460}{z^4}\right]$

Table 4: Total Membrane Stiffness K_{mT} Equation for Six Plate Types

Plate Type	$K_{mT} = \left[k_{mx} + \frac{2k_{mxy}}{Ƨ^2} + \frac{k_{my}}{Ƨ^4}\right]$
CCCC	$K_{mT} = \left[0.0000000024 + \frac{0.0000000010}{Ƨ^2} + \frac{0.0000000024}{Ƨ^4}\right]$
CSSS	$K_{mT} = \left[0.0000340978 + \frac{0.0000091499}{Ƨ^2} + \frac{0.0000449254}{Ƨ^4}\right]$
CSCS	$K_{mT} = \left[0.0000016447 + \frac{0.0000005219}{Ƨ^2} + \frac{0.0000019245}{Ƨ}\right]$
CCSS	$K_{mT} = \left[0.0000011786 + \frac{0.0000003029}{Ƨ^2} + \frac{0.0000011786}{Ƨ^4}\right]$
CCCS	$K_{mT} = \left[0.0000000568 + \frac{0.0000000173}{Ƨ^2} + \frac{0.0000000505}{Ƨ^4}\right]$

Table 5: Buckling/Postbuckling Load and Stress Equations for the Various Plate Types

Plate Type	$N_x = \eta_w \frac{D}{a^2}; \quad \eta_w = \left[\frac{k_{bT}}{k_{Nx}} + \frac{3}{2}\frac{k_{mT}}{k_{Nx}}\frac{1}{(h_{max})^2}\left(\frac{w}{t}\right)^2\right]; \quad \sigma_x = \eta_w \frac{D}{ta^2}$
CCCC	$N_x = \frac{1}{Ƨ^4}\left[\{42(Ƨ^4+1) + 24Ƨ^2\} + \{7.9178674064(Ƨ^4+1) + 3.2048510943Ƨ^2\}\left(\frac{w}{t}\right)^2\right]\frac{D}{a^2}$
	$\sigma_x = \frac{1}{Ƨ^4}\left[\{42(Ƨ^4+1) + 24Ƨ^2\} + \{7.9178674064(Ƨ^4+1) + 3.2048510943Ƨ^2\}\left(\frac{w}{t}\right)^2\right]\frac{D}{ta^2}$
CSSS	$N_x = \frac{1}{Ƨ^4}[(9.8823529412Ƨ^4 + 22.7368421053Ƨ^2 + 24.1857585139) + (9.1530030681Ƨ^4 + 2.4561388409Ƨ^2 + 12.0594968822)\left(\frac{w}{t}\right)^2]\frac{D}{a^2}$
	$\sigma_x = \frac{1}{Ƨ^4}[(9.8823529412Ƨ^4 + 22.7368421053Ƨ^2 + 24.1857585139) + (9.1530030681Ƨ^4 + 2.4561388409Ƨ^2 + 12.0594968822)\left(\frac{w}{t}\right)^2]\frac{D}{ta^2}$
CSCS	$N_x = \frac{1}{Ƨ^4}[(9.8823529412Ƨ^4 + 24Ƨ^2 + 51.0588235294) + (8.3882818821Ƨ^4 + 2.6619116143Ƨ^2 + 9.8154941152)\left(\frac{w}{t}\right)^2]\frac{D}{a^2}$
	$\sigma_x = \frac{1}{Ƨ^4}[(9.8823529412Ƨ^4 + 24Ƨ^2 + 51.0588235294) + (8.3882818821Ƨ^4 + 2.6619116143Ƨ^2 + 9.8154941152)\left(\frac{w}{t}\right)^2]\frac{D}{ta^2}$
CCSS	$N_x = \frac{1}{Ƨ^4}[(21(Ƨ^4+1) + 22.7368421053Ƨ^2) + (11.2046149093(Ƨ^4+1) + 2.8800996574Ƨ^2)\left(\frac{w}{t}\right)^2]\frac{D}{a^2}$
	$\sigma_x = \frac{1}{Ƨ^4}[(21(Ƨ^4+1) + 22.7368421053Ƨ^2) + (11.2046149093(Ƨ^4+1) + 2.8800996574Ƨ^2)\left(\frac{w}{t}\right)^2]\frac{D}{ta^2}$
CCCS	$N_x = \frac{1}{Ƨ^4}[(21Ƨ^4 + 24Ƨ^2 + 44.3333333333) + (10.2684842931Ƨ^4 + 3.1213914297Ƨ^2 + 9.1196865657)\left(\frac{w}{t}\right)^2]\frac{D}{a^2}$
	$\sigma_x = \frac{1}{Ƨ^4}[(21Ƨ^4 + 24Ƨ^2 + 44.3333333333) + (10.2684842931Ƨ^4 + 3.1213914297Ƨ^2 + 9.1196865657)\left(\frac{w}{t}\right)^2]\frac{D}{ta^2}$

When the aspect ratio, $Ƨ = \frac{b}{a}$, is equal to unity, that is a square plate. Then the mathematical models in Table 5 reduce to those presented in Table 6.

Table 6: Buckling/Postbuckling Load and Stress Equations for the Various Square Plate Types

Plate Type	$N_x = \eta_w \frac{D}{a^2};$	$\sigma_x = \frac{N_x}{t}$
	$\eta_w = \left[\frac{k_{bT}}{k_{Nx}} + \frac{3}{2}\frac{k_{mT}}{k_{Nx}}\frac{1}{(h_{max})^2}\left(\frac{w}{t}\right)^2\right]; \quad \beta = \frac{b}{a} = 1$	
CCCC	$N_x = \left[108 + 19.0406\left(\frac{w}{t}\right)^2\right]\frac{D}{a^2}$	$\sigma_x = \left[108 + 19.0406\left(\frac{w}{t}\right)^2\right]\frac{D}{ta^2}$
CSSS	$N_x = \left[56.8050 + 23.6686\left(\frac{w}{t}\right)^2\right]\frac{D}{a^2}$	$\sigma_x = \left[56.8050 + 23.6686\left(\frac{w}{t}\right)^2\right]\frac{D}{ta^2}$
CSCS	$N_x = \left[84.9412 + 20.8657\left(\frac{w}{t}\right)^2\right]\frac{D}{a^2}$	$\sigma_x = \left[84.9412 + 20.8657\left(\frac{w}{t}\right)^2\right]\frac{D}{ta^2}$
CCSS	$N_x = \left[64.7368 + 25.2893\left(\frac{w}{t}\right)^2\right]\frac{D}{a^2}$	$\sigma_x = \left[64.7368 + 25.2893\left(\frac{w}{t}\right)^2\right]\frac{D}{ta^2}$
CCCS	$N_x = \left[89.3333 + 22.5096\left(\frac{w}{t}\right)^2\right]\frac{D}{a^2}$	$\sigma_x = \left[89.3333 + 22.5096\left(\frac{w}{t}\right)^2\right]\frac{D}{ta^2}$

The new mathematical models in Table 5 are simple to use both for analysis and design. They showed both the expressions for buckling and postbuckling load and stress for the various plate types. Fig. 1 shows the relationship between the stress parameter and the ratio of displacement to the thickness of the plate from Equation 1. And the numerical values obtained from equations in Table 6 for buckling and postbuckling loads are presented in Table 7.

These values will help in analyzing the validity of these new mathematical models. A look at the models indicated that only three parameters are unknown, (ie Displacement, w; plate thickness, t; and the buckling and postbuckling load, N_{xMax}). Knowing any two of the variables the other can be evaluated easily.

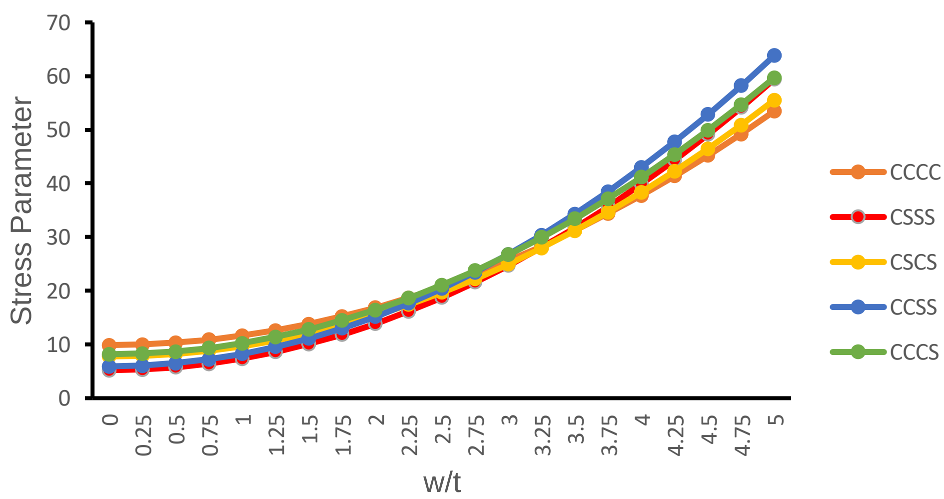


Fig. 1: Stress Parameter ($\frac{\sigma_x a^2}{Et^2}$) vs Displacement to Thickness Ratio (β)

Table 7: Numerical Values of Buckling and Postbuckling Coefficient, η, of Rectangular Plates for Various Square Plate Types.

w/t	$N_x = \eta_w \frac{D}{a^2}$; $2 = \frac{b}{a} = 1$, $\eta_w = \left[\frac{K_{bT}}{k_{Nx}} + \frac{3}{2} \frac{K_{mT}}{k_{Nx}} \frac{1}{(h_{max})^2} \left(\frac{w}{t}\right)^2 \right]$				
	CCCC	CSSS	CSCS	CCSS	CCCS
0	108.000	56.805	84.941	64.737	89.333
0.25	109.190	58.285	86.245	66.317	90.740
0.5	112.760	62.726	90.158	71.059	94.961
0.75	118.710	70.128	96.678	78.962	101.995
1	127.041	80.491	105.807	90.026	111.843
1.25	137.751	93.814	117.544	104.251	124.505
1.5	150.841	110.098	131.889	121.638	139.980
1.75	166.312	129.343	148.843	142.185	158.269
2	184.162	151.549	168.404	165.894	179.372
2.25	204.393	176.715	190.574	192.764	203.288
2.5	227.004	204.842	215.352	222.795	230.018
2.75	251.994	235.930	242.738	255.987	259.562
3	279.365	269.979	272.733	292.341	291.919
3.25	309.116	306.988	305.336	331.855	327.091
3.5	341.247	346.958	340.547	374.531	365.075
3.75	375.758	389.889	378.366	420.368	405.874
4	412.649	435.781	418.793	469.366	449.486
4.25	451.921	484.633	461.829	521.525	495.912
4.5	493.572	536.446	507.473	576.846	545.152
4.75	537.603	591.220	555.725	635.327	597.205
5	584.015	648.954	606.585	696.970	652.072

Both Fig. 1 and Table 7 indicated a gradual increase in postbuckling values of a plate as the displacement-to-thickness ratio increased due to the applied load. This shows that a thin rectangular plate possessed additional strength beyond the critical load or yield point. This property of a thin plate is beneficial in aerospace and naval architecture industries where lightweight structures with higher strength are required for designs. The implication also is that the plate may not fail easily based on geometric conditions but rather it will fail based on material weakness. This is in line with various assertions in literature (Chajes, 1974, Ventsel and Kruamerther, 2001, Oguaghamba, 2015). Since their critical values correspond with those in the literature (Ibearugbulem, 2014; Adah, 2016, Onwuka, et al 2016, Iwuoha, 2016). These results fulfilled the aim of this work which is to formulate adequate and simple specific mathematical models for stability analysis of five thin isotropic rectangular plate types under large deflection.

IV. CONCLUSION

The present work has formulated the nonlinear strain-displacement relations from the first principle and specific mathematical models for stability analysis of thin isotropic rectangular plates under large deflection for five plate types namely, CCCC, CSSS, CSCS, CCSS, and CCCS as presented in Table 5. The numerical results obtained from these models for the various plate types indicated that the formulated models are adequate for the stability analysis of a thin isotropic plate with large deflection. The simplicity of the formulated models is another plus to this work, as this will offer quick results in analysis and design. Therefore, the conclusion is that these newly formulated mathematical models are adequate for the stability analysis of thin isotropic rectangular plates for any plate type and a better alternative to the long-standing von Karmon equations that are limited in application due to the complexity of getting an exact solution.

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